Game Theory

Lecture 9: Dynamic Games and Perfect Equilibrium

Frieder Neunhoeffer

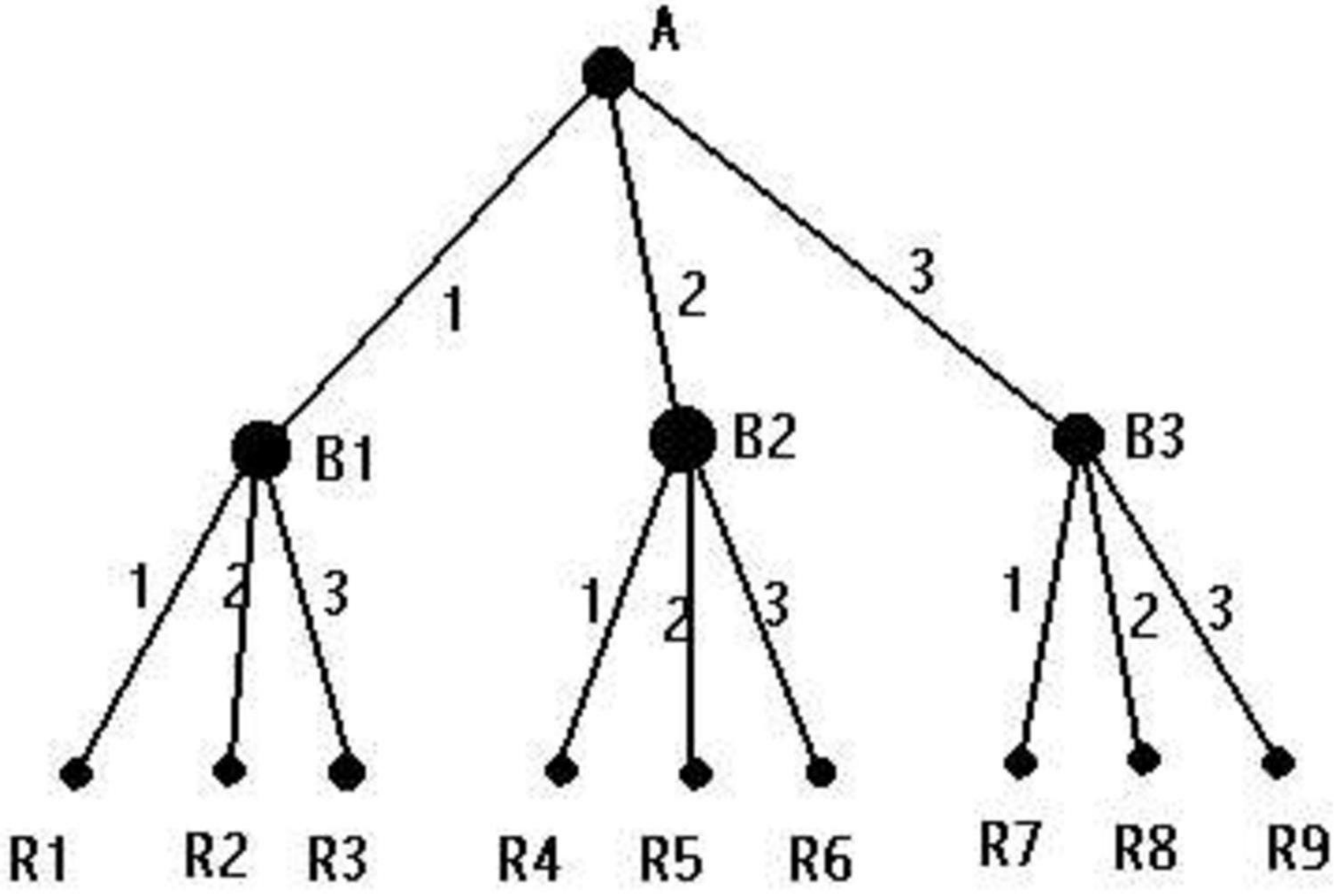
Lisbon School of Economics & Management
Universidade de Lisboa

Representation of dynamic games: the extensive form

In a dynamic game, the participants move *sequentially* rather than *simultaneously*, so that players that move in the last place can observe and react to the choices made by first movers. The adequate representation is the so-called *extensive form*, that is equivalent to drawing a *game tree*, where:

1. Players are exactly identified.
2. For each player the tree specifies:
    - when the player acts, i.e., their decision points
    - their basic alternatives of action in each decision point
    - what the player knows in each decision point
    - the payoff received by each player in each possible game outcome, related with a set of actions chosen by all players

Sequential game in extensive form



The nodes represents the player (*A* or *B*) called to act.

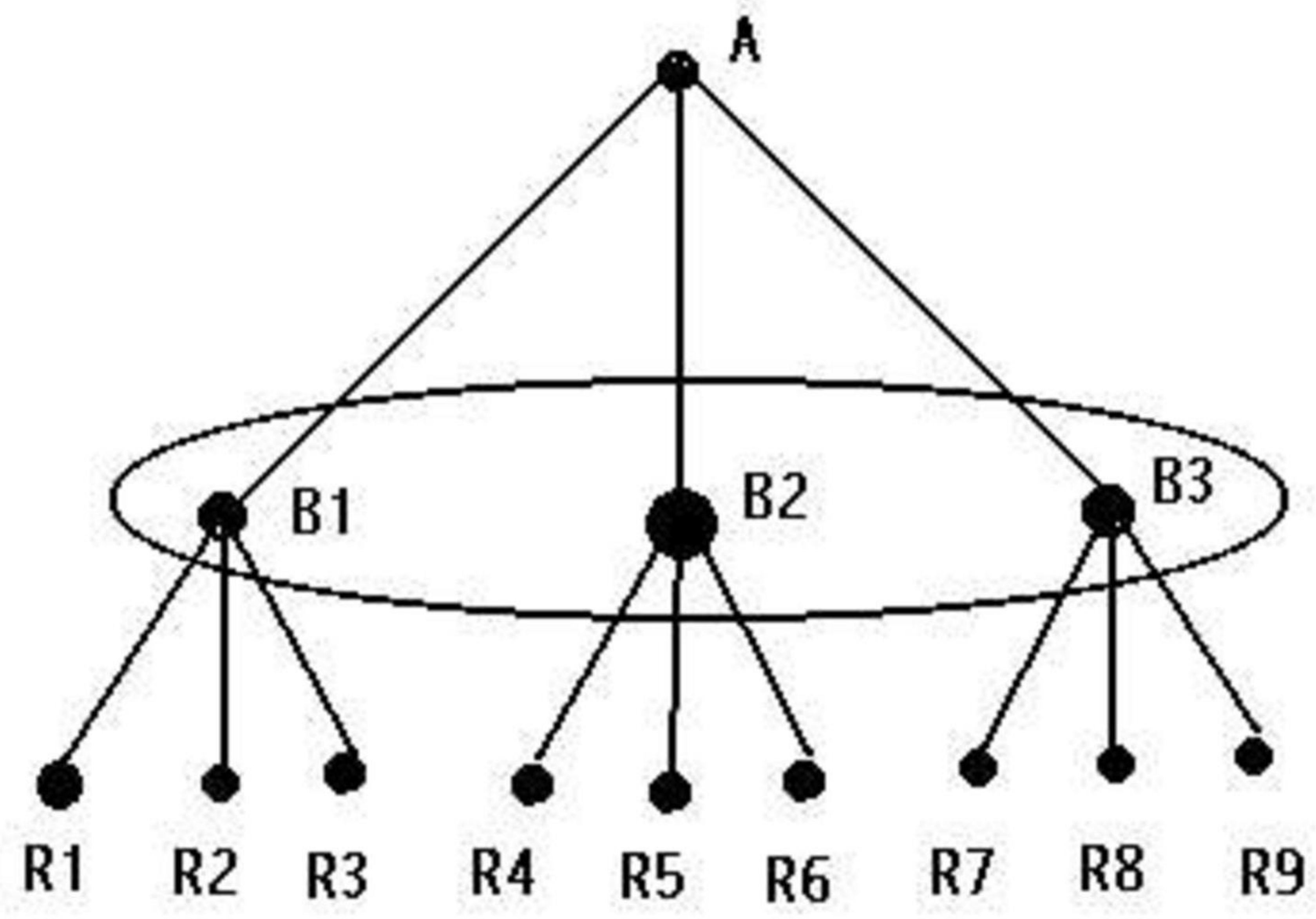
The branches deriving from each node represent the basic alternatives of action available to the player who is identified by the node.

The final nodes represent the possible outcomes/results of the game.

In a dynamic (or sequential) game plotted through the extensive form, a key distinction is the one that opposes the notion of "action or "movement" to the concept of "strategy":

- **An action/movement** is a *simple choice* that the player makes in a certain moment during the gradual development of the game.
- **A player's strategy** is a *comprehensive plan* that prescribes an action in each one of its possible decision points. In other words: a *strategy* is a function that associates to each information set of the player one of the alternatives that follows from this set.

A static game in extensive form



The figure plots a game where the players move *simultaneously*, where each player does *not* observe the action chosen by the opponent. In this game:

The fact that the nodes B_1, B_2 and B_3 are surrounded by an oval means that they belong to the same *information set*: given the fact that the players move simultaneously, *B* can not distinguish between the decision points B_1, B_2 and B_3.

The information set

Definition of information set: It is a collection of *decision nodes* that satisfy two properties:

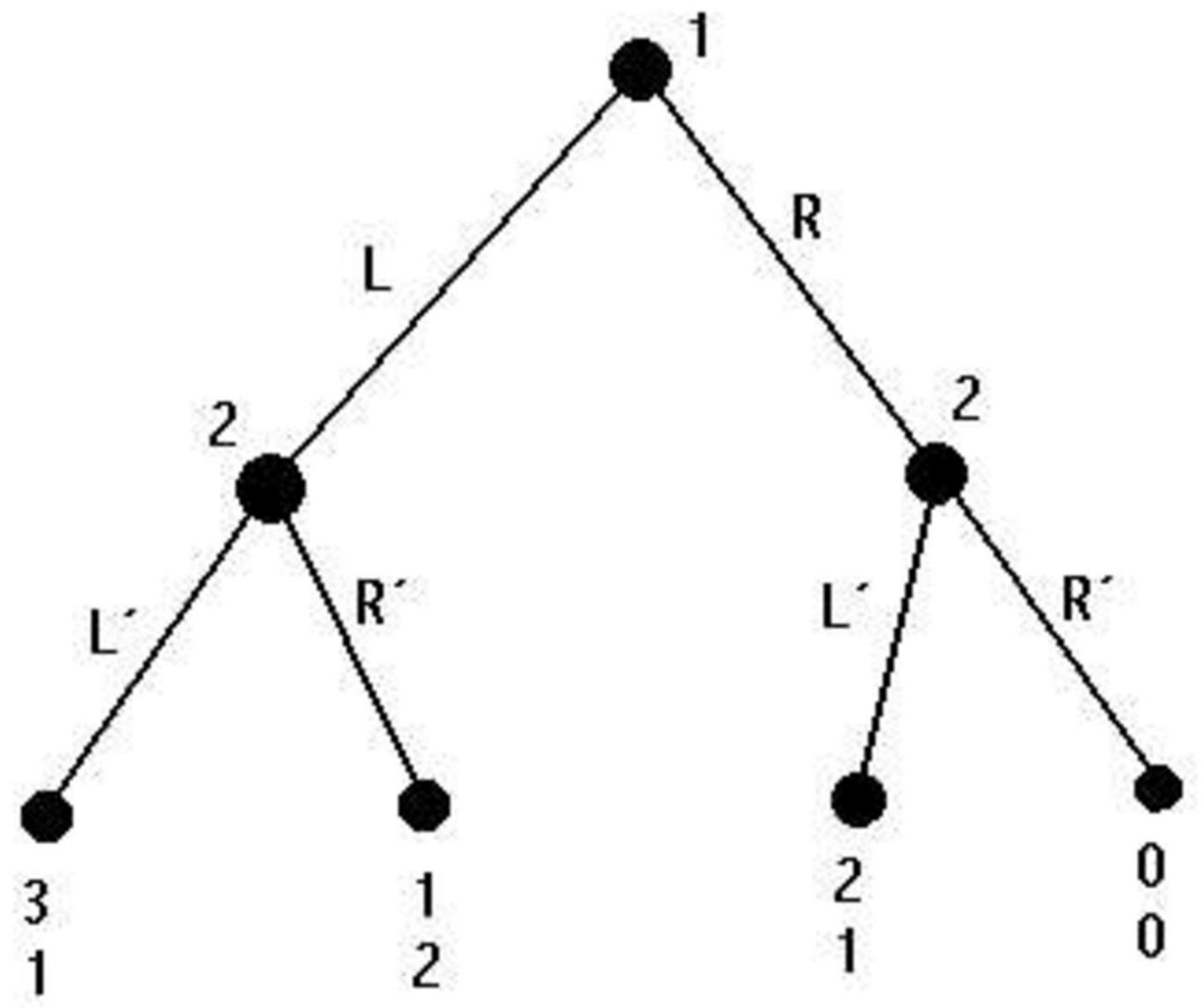
a) The player can act in each node of the information set.

b) When the game reaches a node within the information set, the concerned player *does not know* which node has been effectively reached.

An *information set* should meet the following condition: The player should have the same set of feasible actions in each node within an information set. Otherwise, the player could infer which node had been reached when she is called to act.

Sequential game in extensive form

Another example



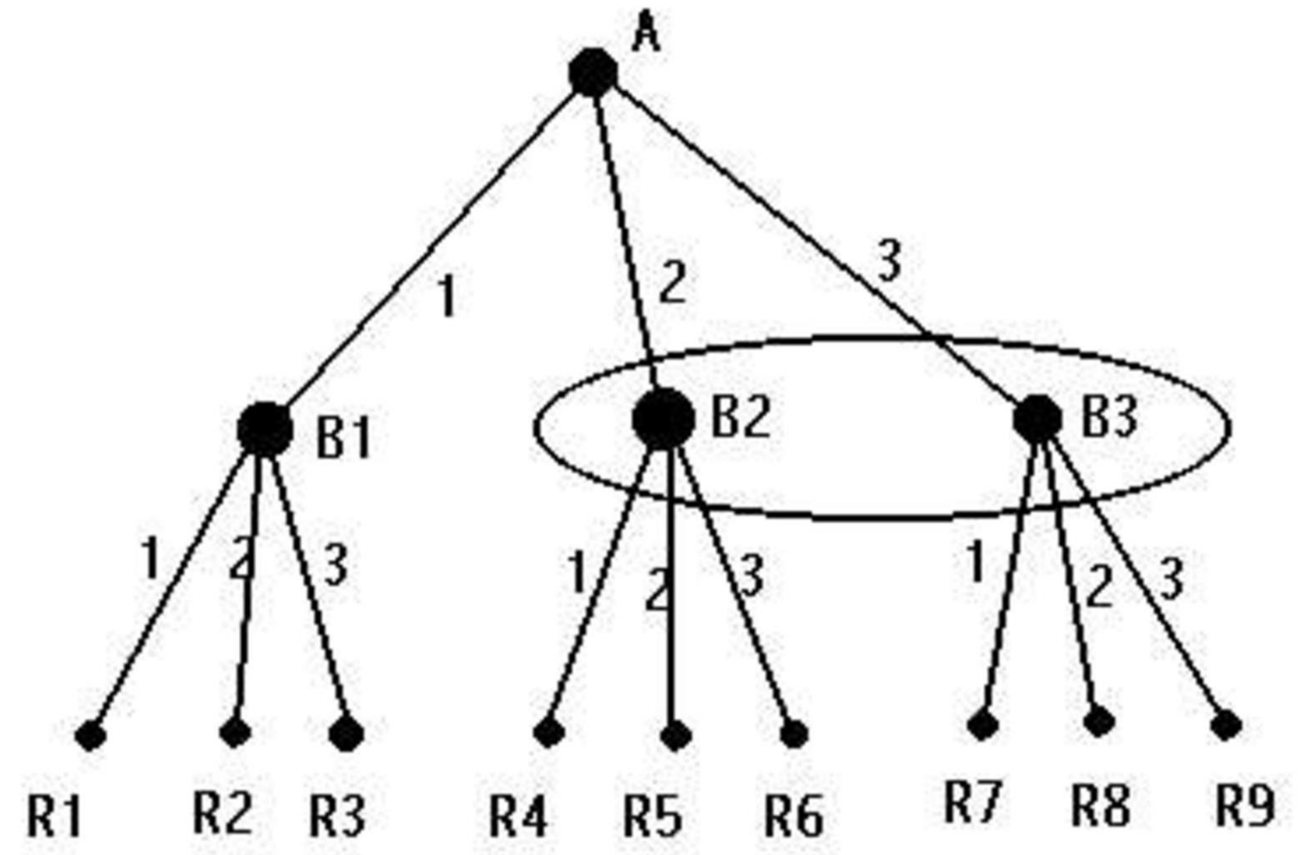
Consider the following game with strategy sets of players being:

- Player 1: $\{L, R\}$.
- Player 2: $\{(L', L'), (R', R'), (L', R'), (R', L')\}$, where the first element in each strategy refers to the action chosen when player 1 played her first strategy and the second element refers to the action chosen regarding player 1 chose her second strategy. For instance, (R', R') labels the strategy "if player 1 selected L before, play R'; if player 1 selected R before, play R'".

Hence, the normal form of the game is:

		Player 2			
		(L', L')	(R', R')	(L', R')	(R', L')
Player 1	L	3,1	1,2	3,1	1,2
	R	2,1	0,0	0,0	2,1

Game with an intermediate information level



The size of a player's strategy set depends critically on the order of moves and on the information structure. In the game plotted in Slide 4, each player has three strategies which are coincident with the actions.

By contrast, in the game plotted in Slide 3, player *A* has again three strategies, but player *B* can select one of $3 \times 3 \times 3 = 27$ strategies.

We can model a game that is intermediate in relation to both games. In this game, player *B* has enough information to distinguish the choice of strategy 1 by player *A* from the choice of either strategy *2* or strategy *3*, but her information is insufficient to determine whether *A* has selected strategy *2* or strategy *3*.

In this game, player *A* keeps her three strategies, but player *B* has nine strategies of the following type:

If *A* plays 1, *B* replies with $i = 1,2,3$.

If *A* plays 2 or 3, *B* replies with $j = 1,2,3$.

Solution concept in static games: perfect equilibrium

We have seen that, in static games, sometimes multiple Nash equilibria arise. The concept of Nash *perfect* equilibrium allows us to select the more robust among several Nash equilibria that emerge. To clarify the concept of *perfect equilibrium*, we introduce the following static game.

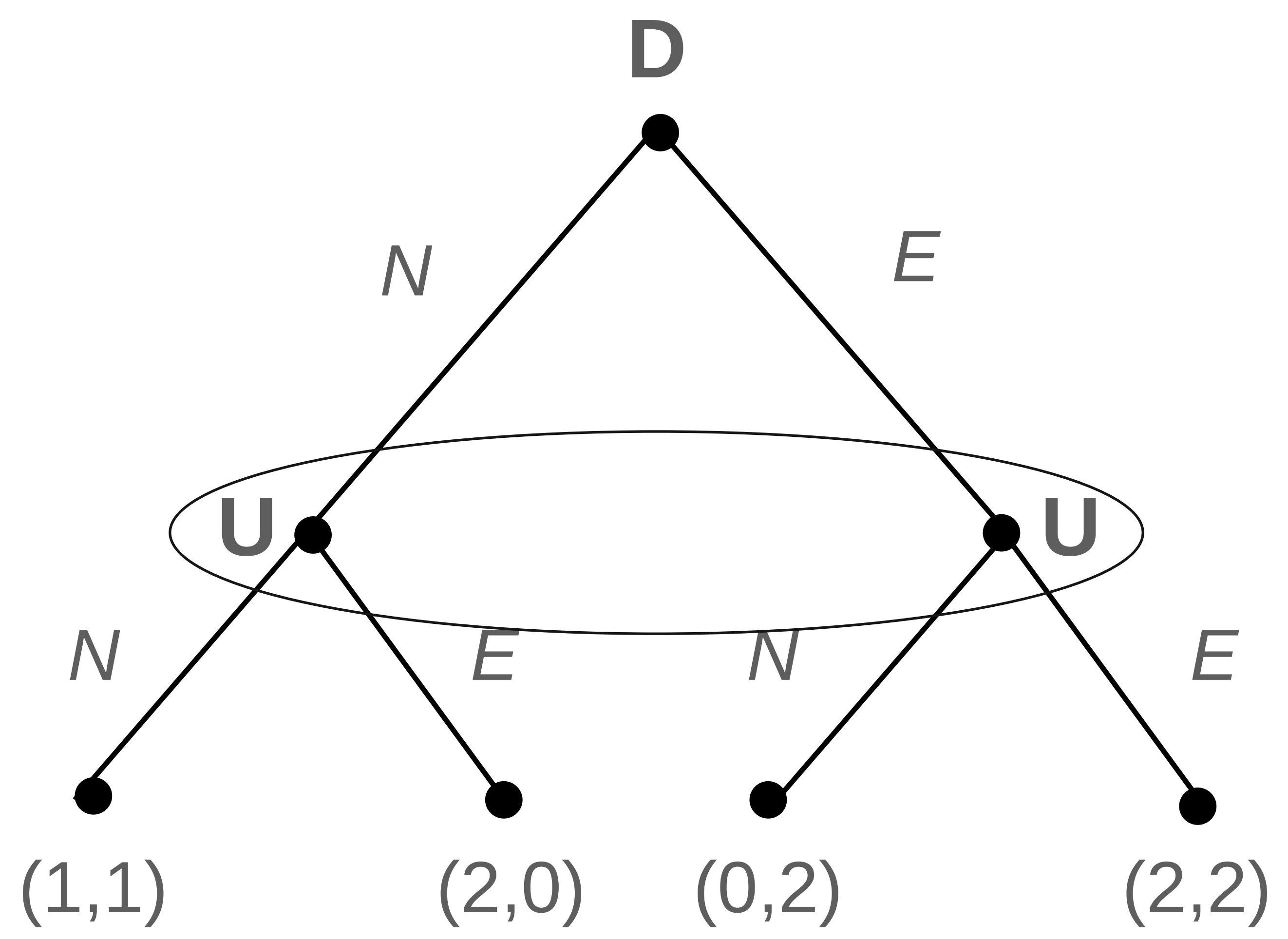
Game description:
In an industry, two vertically-linked firms coexist. An upstream firm *U* manufactures a component that is finished and distributed by a downstream firm *D*. Suppose that *D* plans to enter a new market. However, this strategy is successful only if firm *U* decides to enter as well. Hence, each firm has two strategies: either to not enter (strategy N) or to enter (strategy E). If neither firm enters, both firms get 1 each. If both enter, each one gets a profit of 2. If only one enters, the firm entering gets 0 and the other gets 2. The extensive and normal forms of the game are depicted on the next slide.

Static coordination game

Normal form:

		U	
		N	E
D	N	1,1	2,0
	E	0,2	2,2

Extensive form:



Normal-form “trembling-hand" perfect equilibrium

This is a *symmetric game*. It is clear from the payoff matrix that there are two Nash equilibria in pure strategies, namely (E, E), with payoffs $(2{,}2)$, and (N, N), with payoffs $(1{,}1)$. There is no equilibrium in completely mixed strategies.

How to select the "good/perfect" Nash equilibrium?

A first answer would be to assume is the one that survives the introduction of an arbitrarily small probability of error $\epsilon > 0$ when each player selects her pure strategy, as if she had a "trembling hand".

To be more specific, we replace each pure strategy by a **mixed** strategy, where the player selects the intended pure strategy with probability $(1 - \epsilon)$, but can by mistake select the "other" pure strategy with probability ϵ. It is then easy to calculate the expected payoffs of the players with these mixed strategies. Since the game is symmetric, we will only account for the payoffs of player *D*.

Payoffs

To compute the first cell (a_{11}) in the payoff matrix, pure strategy N will be replaced by the mixed strategy $(1-\epsilon, \epsilon)$ while strategy E will be substituted by $(\epsilon, 1-\epsilon)$. The respective expected payoff become (the terms with ϵ^2 are so small that they are set equal to zero):

$$a_{11} = (1-\epsilon, \epsilon)\begin{pmatrix}1 & 2\\ 0 & 2\end{pmatrix}\begin{pmatrix}1-\epsilon\\ \epsilon\end{pmatrix} = 1+\epsilon^2 \approx 1$$

$$a_{12} = (1-\epsilon, \epsilon)\begin{pmatrix}1 & 2\\ 0 & 2\end{pmatrix}\begin{pmatrix}\epsilon\\ 1-\epsilon\end{pmatrix} = 2-\epsilon-\epsilon^2 \approx 2-\epsilon$$

$$a_{21} = (\epsilon, 1-\epsilon)\begin{pmatrix}1 & 2\\ 0 & 2\end{pmatrix}\begin{pmatrix}1-\epsilon\\ \epsilon\end{pmatrix} = 3\epsilon-\epsilon^2 \approx 3\epsilon$$

$$a_{22} = (\epsilon, 1-\epsilon)\begin{pmatrix}1 & 2\\ 0 & 2\end{pmatrix}\begin{pmatrix}\epsilon\\ 1-\epsilon\end{pmatrix} = 2-2\epsilon+\epsilon^2 \approx 2-2\epsilon$$

Perturbed game $G(\epsilon)$

Hence, the perturbed game $G(\epsilon)$ is given by:

$$G(\epsilon) = \begin{array}{cccc} & & \boldsymbol{U} & \\ & & N & E \\ \boldsymbol{D} & N & 1,1 & 2-\epsilon, 3\epsilon \\ & E & 3\epsilon, 2-\epsilon & 2-2\epsilon, 2-2\epsilon \end{array}$$

Consider now a decreasing sequence $\{\epsilon_i\}, i = 1,2,3, \ldots$ that converges to 0. Two related sequences arise:

1. The sequence of perturbed games $\{G(\epsilon_i)\}, i = 1,2,3, \ldots$ where $G(\epsilon_i)$ is the game perturbed according to error term ϵ_i.

2. The sequence of sets $\{\Theta_i^{NE}\}, i = 1,2,3, \ldots$ where Θ_i^{NE} is the set of Nash equilibria in perturbed game $G(\epsilon_i)$.

It is clear that, when ϵ is arbitrarily close to 0 (but still positive), there is a NE in the perturbed game $G(\epsilon)$ that is (N, N), while (E, E) is no longer a NE in the perturbed game. Consequently, (N, N) is the only "perfect" Nash equilibrium in the base game.

Solution concept in dynamic games: perfect equilibrium

Now consider the following sequential game.

Game description:
In an industry, two vertically-linked firms coexist. An upstream firm *U* manufactures a component that is finished and distributed by a downstream firm *D*. Suppose that *D* plans to manufacture a new product. However, this product innovation is fully successful only if firm *U* decides to produce and sell a new and fit input. Hence, each firm has two strategies: either to invest (strategy I) or not to invest (strategy NI). If both invest, each one gets a profit of 4. Otherwise, each one has zero profits. We assume that firm *D* moves first. The extensive and normal forms of the game are depicted on the next slide.

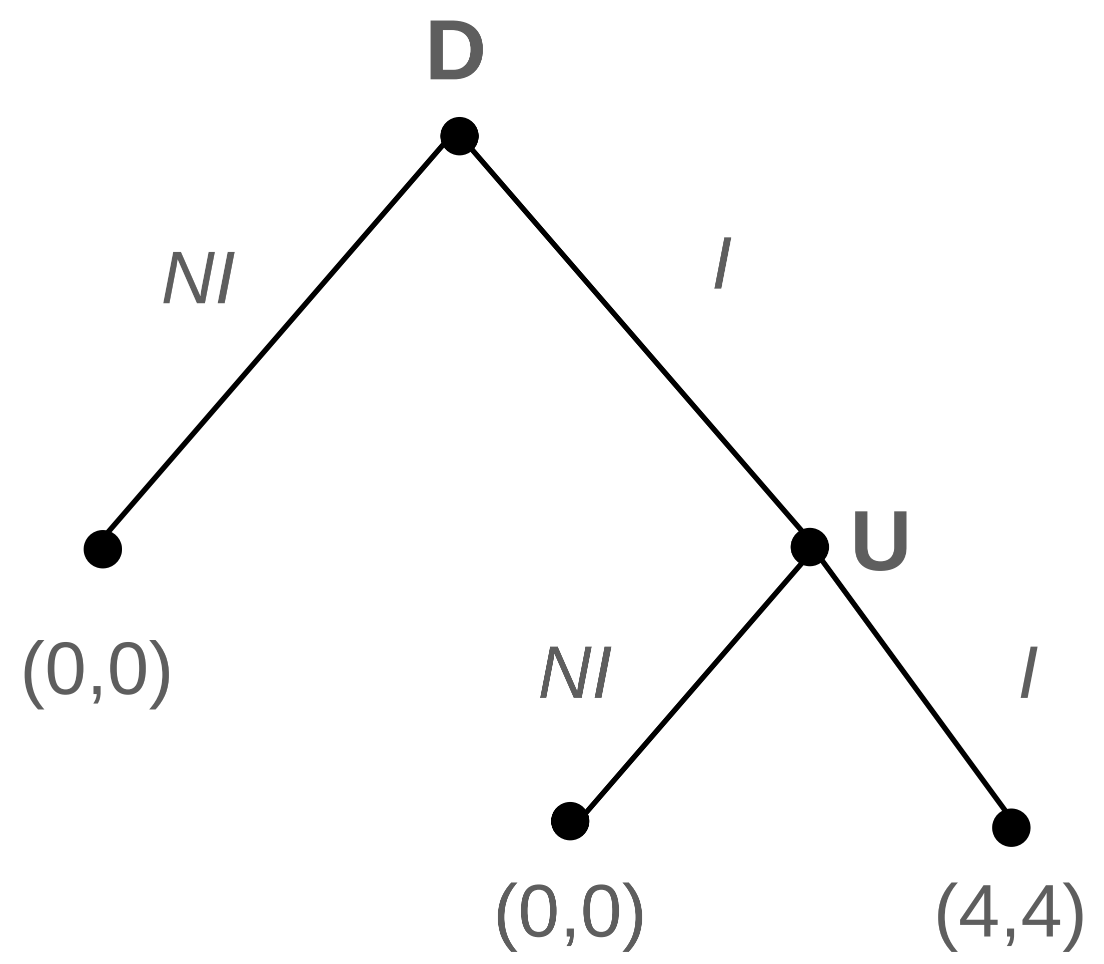
Remark: We can see on the next slide that the same normal form can have more than one corresponding extensive form (compare with slide 9). Hence, one can say that the extensive form of a game carries more information than the normal form.

Dynamic coordination game

Normal form:

		U	
		NI	I
D	NI	0,0	0,0
	I	0,0	4,4

Extensive form:



Payoffs

Let's apply again the *trembling-hand perfect equilibrium concept*. Computing the first cell (a_{11}) in the payoff matrix of player *D*, pure strategy NI will be replaced by the mixed strategy $(1-\epsilon,\epsilon)$ while strategy I will be substituted by $(\epsilon, 1-\epsilon)$. The respective expected payoffs become (terms with ϵ^2 are so small that they are set equal to zero):

$$a_{11} = (1-\epsilon,\epsilon)\begin{pmatrix}0 & 0\\0 & 4\end{pmatrix}\begin{pmatrix}1-\epsilon\\ \epsilon\end{pmatrix} = 4\epsilon^2 \approx 0$$

$$a_{12} = (1-\epsilon,\epsilon)\begin{pmatrix}0 & 0\\0 & 4\end{pmatrix}\begin{pmatrix}\epsilon\\ 1-\epsilon\end{pmatrix} = 4\epsilon - 4\epsilon^2 \approx 4\epsilon$$

$$a_{21} = (\epsilon,1-\epsilon)\begin{pmatrix}0 & 0\\0 & 4\end{pmatrix}\begin{pmatrix}1-\epsilon\\ \epsilon\end{pmatrix} = 4\epsilon - 4\epsilon^2 \approx 4\epsilon$$

$$a_{22} = (\epsilon,1-\epsilon)\begin{pmatrix}0 & 0\\0 & 4\end{pmatrix}\begin{pmatrix}\epsilon\\ 1-\epsilon\end{pmatrix} = 4 - 8\epsilon + 4\epsilon^2 \approx 4-8\epsilon$$

Perturbed game $G(\epsilon)$

Hence, the perturbed game $G(\epsilon)$ is given by:

$$G(\epsilon) = \begin{array}{ccc|c|c} & & & \boldsymbol{D} & \\ & & & NI & I \\ \boldsymbol{U} & NI & & 0,0 & 4\epsilon, 4\epsilon \\ & I & & 4\epsilon, 4\epsilon & 4-8\epsilon, 4-8\epsilon \end{array}$$

It is clear that, when ϵ is arbitrarily close to 0 (but still positive), there is a NE in the perturbed game $G(\epsilon)$ that is (I, I), while (NI, NI) is no longer a NE in the perturbed game. Consequently, (I, I) is the only "perfect" Nash equilibrium in the base game.

Solution concept in dynamic games: perfect equilibrium

Now consider the following <u>sequential</u> game.

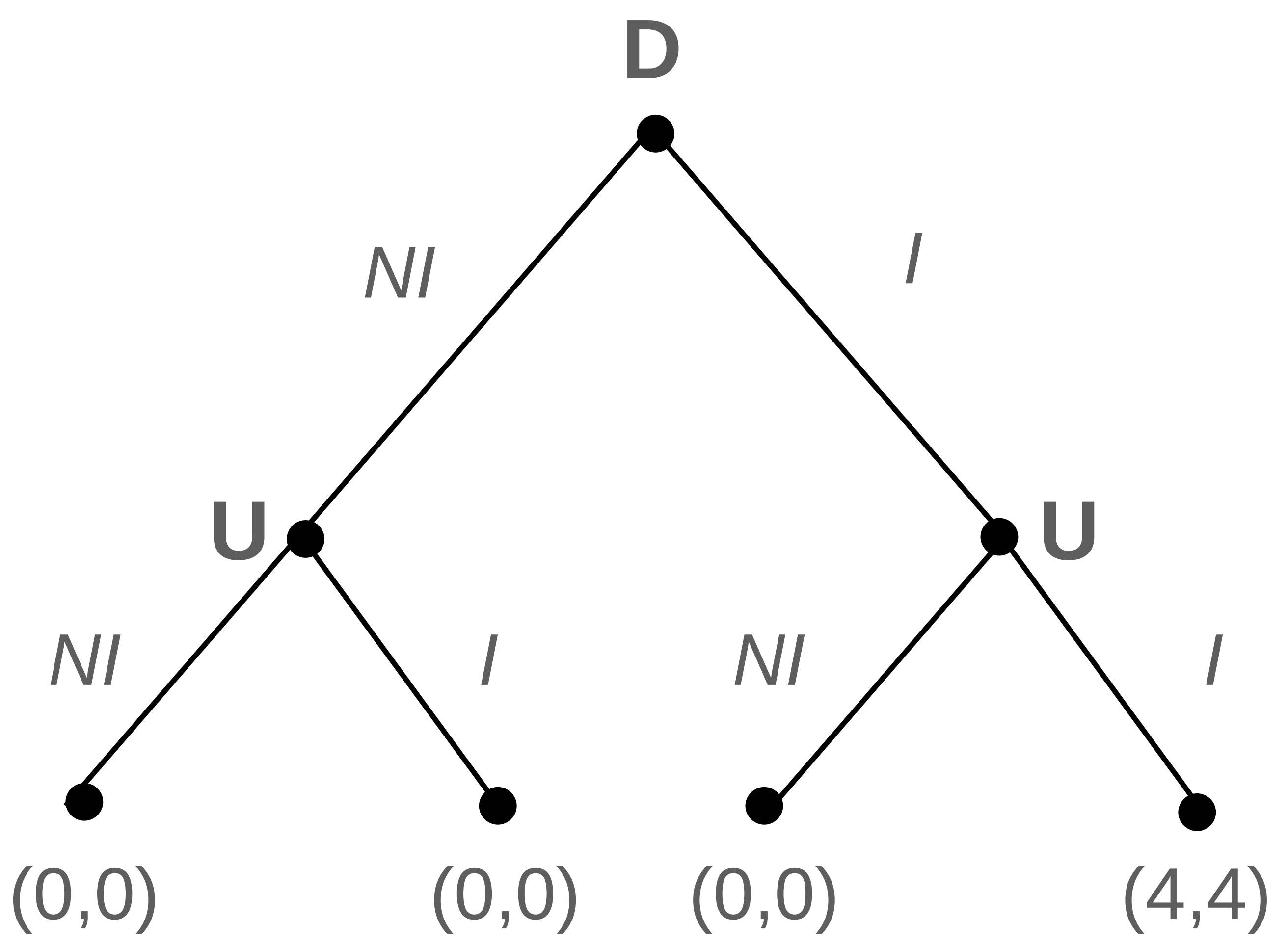
Game description:
In an industry, two vertically-linked firms coexist. An upstream firm *U* manufactures a component that is finished and distributed by a downstream firm *D*. Suppose that *D* plans to manufacture a new product. However, this product innovation is fully successful only if firm *U* decides to produce and sell a new and fit input. Hence, each firm has two strategies: either to invest (strategy I) or not to invest (strategy NI). If both invest, each one gets a profit of 4. Otherwise, each one has zero profits. If one firm invests and the other doesn't, the investment is not lost but doesn't yield higher revenues than the costs from the investment. We assume that firm *D* moves first. The extensive and normal forms of the game are depicted on the next slide.

Dynamic coordination game

Normal form:

			U		
		NI/NI	NI/I	I/NI	I/I
D	NI	0,0	0,0	0,0	0,0
	I	0,0	4,4	0,0	4,4

Extensive form:



Perfect equilibrium

Proposition: In each game with a finite number of agents, each agent being endowed with a finite number of strategies, there is always at least one perfect equilibrium (either in pure or in mixed strategies). An immediate consequence of this is:

Corollary: If a finite game has a unique Nash equilibrium, then this equilibrium is necessarily perfect.

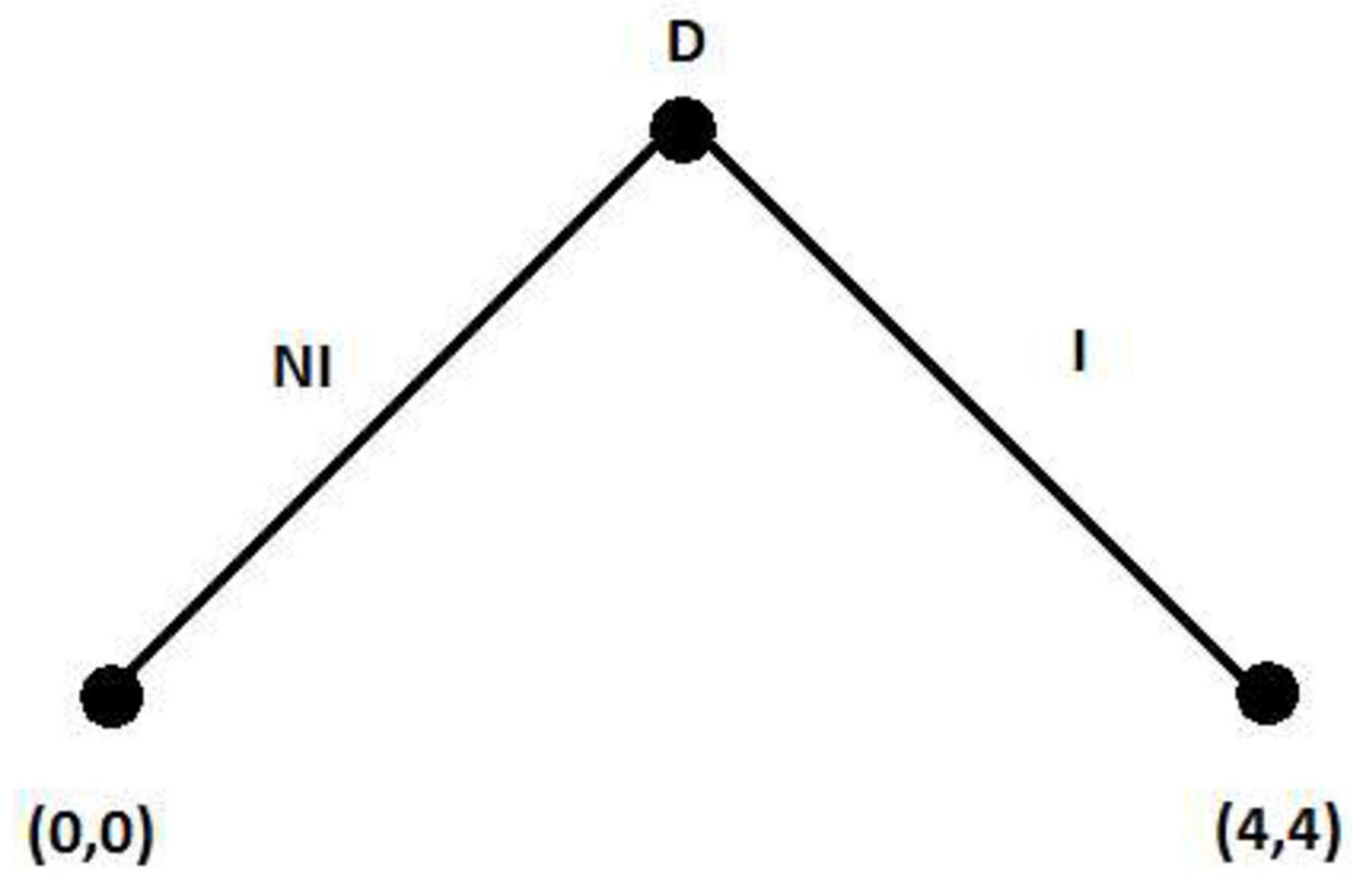
The computation of a perfect equilibrium is usually made through the extensive form. An important definition is:

<u>**Game with perfect information**</u>**:** is a game where, in the extensive form, each information set consists of a single decision node, i.e., the player in each of their decision points has full knowledge of past moves made by all the players. In these games, there are no simultaneous moves. All moves are sequential: Player 1 moves, then Player 2 moves, and so on ...

Games with perfect information

In perfect information games, perfect equilibria can be easily computed through *"backward induction".* We determine first the rational decision by the last player to move and substitute payoffs in her decision node, then we proceed to the immediately preceding player, and so on, until the decision of the first player to act is modeled.

E.g., for the game depicted in Slide 19, we first examine the choice by player *U* given *D*'s choice: if *D* chooses NI player *U* gets payoff 0 regardless of her choice; if *D* chooses I player *U* gets payoff 0 by choosing NI and otherwise 4. Hence, her rational decision is to select I given *D* chooses I so that both players end up with payoff 4, otherwise she is indifferent. If we substitute these payoffs in the decision node of player *U*, we obtain the reduced tree.



Then, it's clear that the rational choice by player *D* is I,
so that the two perfect equilibria are $\big(I,(NI,I)\big)$ and $\big(I,(I,I)\big)$.

“Remaining/Continuation" game

Another way of defining the perfect equilibrium is through the concept of the **"remaining game"** or **"continuation" game**, which is the part of the game tree formed by a decision node and by the "branches" and "nodes" that succeed to it until the game ends.

In the game tree in slide 14, there are two "remaining/continuation" games: the whole game starting in node *D* and the game starting in node *U*. The outcome (I, I) is a perfect equilibrium because in both “continuation“ games it leads to a Nash equilibrium.

Remark: The "remaining/continuation" game starting in node *U* is a *one-person* game, equivalent to an individual decision. In such a game, the Nash equilibrium consists in a *payoff maximizing choice*, from which the player has no incentive to deviate.

Generalization to games with imperfect information

The subgame perfect equilibrium

Generally, in games of *imperfect information*, there are both simultaneous and sequential moves, so a player may not know everything about past moves made by other players. This is expressed by the fact that a decision point may be an information set with several decision nodes.

The main concept is now the notion of *subgame* that generalizes the concept of "remaining/continuation" game, by adding the requirement that the subgame should start with a player endowed with perfect information and that perfect information position should be conserved until the game ends. A formal definition is:

Subgame is a part of the game made up by a node that is also an information set and the branches and nodes that succeed to it, under the condition that any node of the subgame should not belong to an information set containing nodes which do not belong to the subgame.

Subgames in the previous slides

Slide 3: there are four subgames.

Slide 4: there is a unique subgame (i.e., the base game itself).

Slide 6: there are three subgames.

Slide 7: there are two subgames.

Slide 9: there is a unique subgame (i.e., the base game itself).

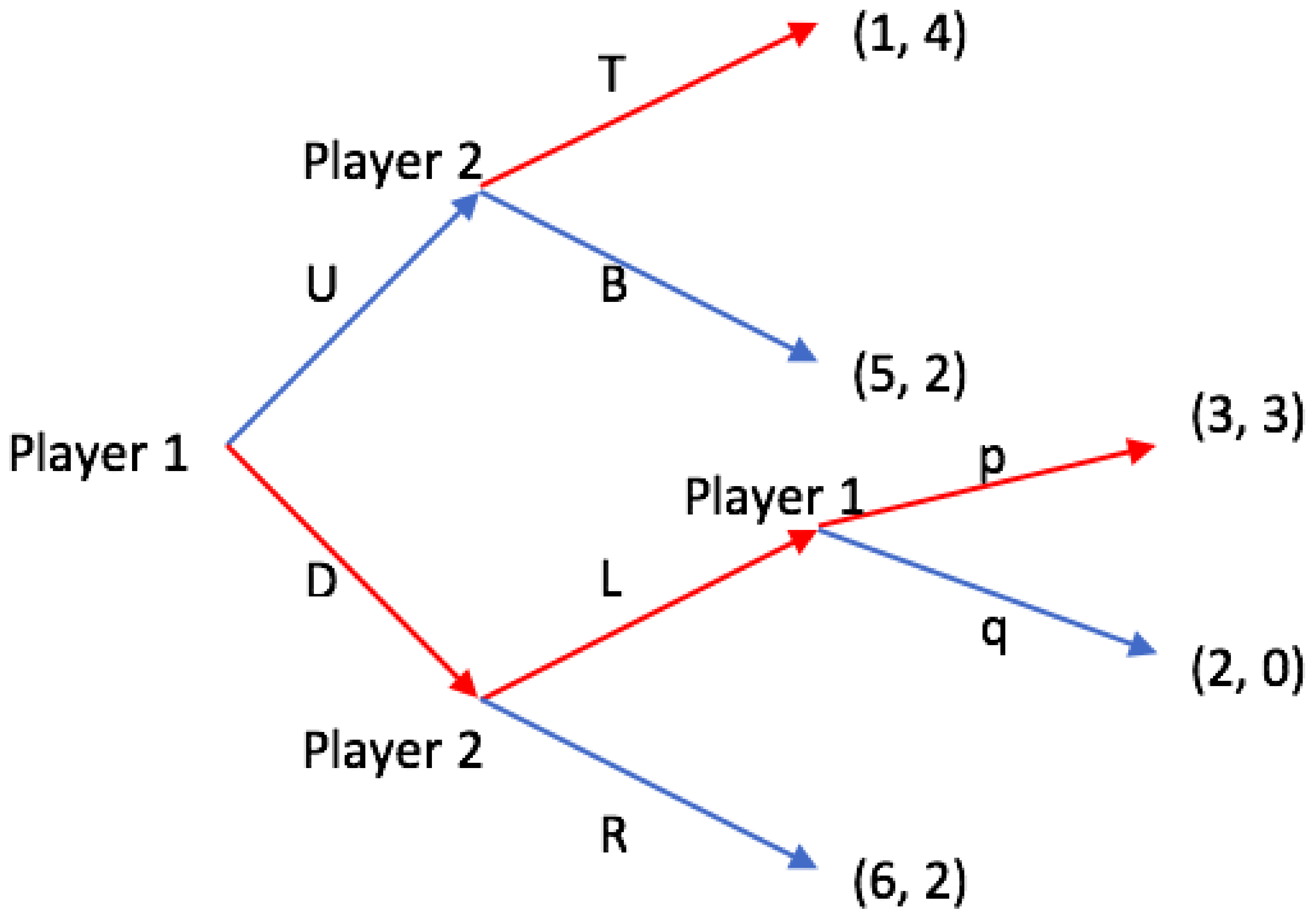
Slide 14: there are two subgames.

Slide 14: there are three subgames.

The definition of *subgame perfect equilibrium* generalizes the former definition of a trembling-hand perfect equilibrium:

Subgame perfect equilibrium: A subgame perfect equilibrium is an equilibrium profile of strategies such that the restriction of these strategies to each subgame generates in this subgame a Nash equilibrium.

Example



Strategies for Player 1 are: Up, Uq, Dp, Dq

Strategies for Player 2 are: TL, TR, BL, BR

There are 4 subgames, with 3 proper subgames.

<u>Subgame for actions p and q</u>: Player 1 will take action p with payoff (3,3), so the payoff for action L becomes (3,3).

<u>Subgame for actions L and R</u>: Player 2 will take action L for 3 > 2, so the payoff for action D becomes (3, 3).

<u>Subgame for actions T and B</u>: Player 2 will take action T to maximize Player 2's payoff, so the payoff for action U becomes (1,4).

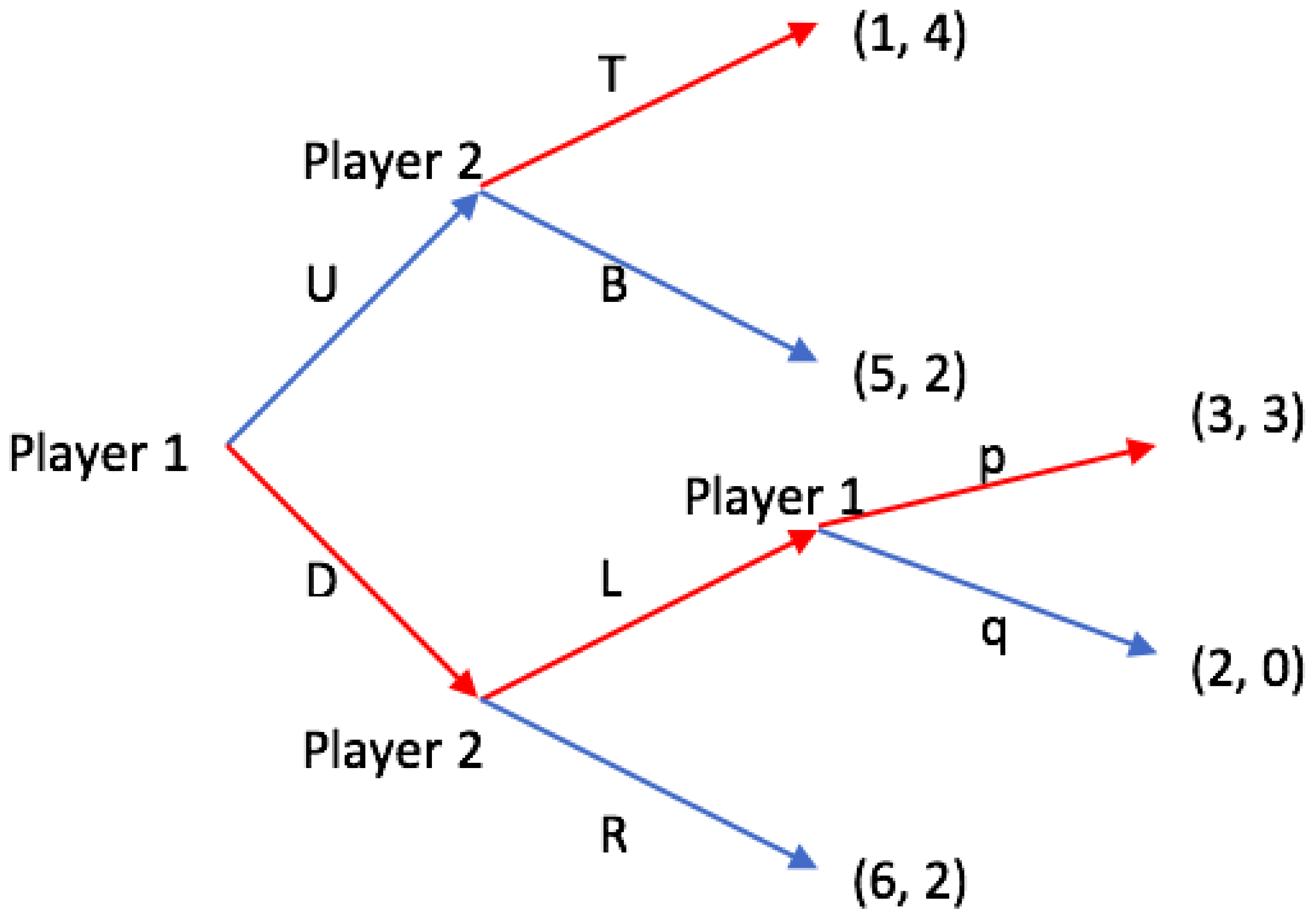
<u>Subgame for actions U and D</u>: Player 1 will take action D to maximize Player 1's payoff.

Example

Thus, the subgame perfect equilibrium is {Dp, TL} with the payoff (3,3).

Remember that it does not matter for the subgame perfect equilibrium, whether a decision node will be effectively reached or not (i.e., subgame for actions T and B).

It ensures a Nash equilibrium in every possible subgame!!!



References

- GIBBONS, Robert (1992), *A primer in game theory*, New York, Harvester Wheatseaf (Chapter 2, Sections 2.1.A. 2.4.A. and 2.4.B, pages 221-231 and 267-282)
- MAS-COLLEL, A. ET AL. (1995), *Microeconomic Theory*, Oxford University Press, (Chapter 7, Sections 7.C, 7.D; Chapter 9, Sections 9.A, 9.B; pages 221-231, 267-282 and 258-260).
- WEIBULL, Jörgen W. (1997), Evolutionary Game Theory, Cambridge Mass./London, England, chapter 1: 15-17.